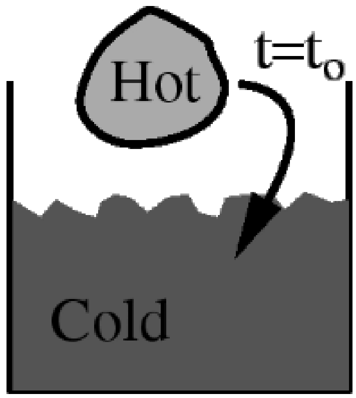
Elementary Non-Steady Phenomena

Because Transport deals with *rates* it is often the case that we must consider non-steady (or transient) operation (when the rates do not exactly cancel). In this section, we examine *scalar* transport (heat and mass transfer) where some non-steady problems can be simplified significantly.

- Explain the utility of the Biot number [18.1]
- Identify "regimes" of transient response based on the value of the Biot number
- Use the "lumped" equation to solve "1D" transient heat transfer problems [18.1]
- Use the "lumped" equation to solve "1D" transient mass transfer problems [27.1]
- Use Gurney-Lurie charts to solve heat and mass transfer problems [18.2, 27.4]
- Use the semi-infinite approximation to solve both transient heat and mass problems for "short" times [18.1, 27.2]

T: Transient Conduction - A "Lumped" Approach

Consider a "hot" piece of iron that we throw into a "cold" oil bath. Our problem is greatly simplified if we assume that the *internal* resistance is low (relative to what?!).



What will be true in this case?

- the temperature is constant (in time) at the boundaries
- the temperature within the solid is spatially uniform
- the flux out of the solid is constant (in time)
- there is no flux out of the solid

So, what we are doing here is making an assumption that makes the problem easier, because the full problem (where the temperature varies in both space *and* time) is too hard (don't worry, we will do the hard version too!).

First, we need to figure out what our governing equation is...

rate of change of heat in the material = net rate of heat flow *into* the material

or in equation form

$$\frac{d(\rho c V T)}{dt} = \rho c V \frac{dT}{dt} = -hA(T - T_\infty)$$

We started looking at this problem by making the assumption that *internal* resistance is negligible. We will now look at the question of: how do we validate this assumption? and: what are the other cases that may present themselves if this *isn't* true?

Early in the course, we spent a lot of time considering thermal resistances. What do we mean by *internal* resistance in this case?!

Conduction, of course! And what is the resistance due to conduction?

$$R_{cond} = R_{internal} = \frac{L}{kA}$$

NOTE:

It is important to note, that we need to be careful hear in our choice of L. A good rule of thumb is that L should be the ratio of V/A.

So,

$$R_{internal} = \frac{V/A}{kA} = \frac{L}{kA}$$

In contrast to *internal* resistance, what *other* resistance is important?

External. And in this case, what is the external resistance?

Convection, of course! And what is the resistance due to convection?

$$R_{conv} = R_{external} = \frac{1}{hA}$$

Finally, how do we *determine* the importance of the internal resistance? By comparing it to the external resistance! This yields a new dimensionless quantity, the Biot (*Bi*) number.

$$Bi = \frac{R_{internal}}{R_{external}} = \frac{\frac{V/A}{kA}}{\frac{1}{hA}} = \frac{Vh/A}{k} = \frac{hL}{k}$$

For internal resistance to be negligible what does the *Bi* have to be like? *How* small?

A good rule of thumb is that for *Bi*<0.1, most geometries yield a *full* solution where the center-line temperature is less than 5% different from the surface temperature (in other words a "lumped" solution would be pretty good). Before we move on to looking at $Bi/ge0.1$, how would this problem change is we had radiation occurring?

If we had radiation, we would *first* have to recalculate our *Bi*. Since radiation is *also* an external resistance, and it is in *parallel* with the convection, we would modify our external resistance to be

$$\frac{1}{R_{external}} = \frac{1}{\frac{1}{ha}} + \frac{1}{\frac{1}{h_r A}} = (h + h_r)A$$

or

$$R_{external} = \frac{1}{(h + h_r)A}$$

so our *Bi* is

$$Bi = \frac{(h + h_r)L}{k}$$

If we find that this *Bi* is sufficiently small, we then need to modify our governing equation. If the temperature difference is sufficiently small (so that our h_r approximation works), we get

$$\rho c V \frac{dT}{dt} = -(h + h_r)A(T - T_\infty)$$

If, however, the temperature difference is *large* (so h_r is not valid), we get something similar to

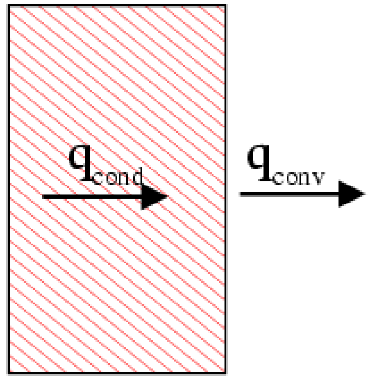
$$\rho c V \frac{dT}{dt} = -hA(T - T_\infty) - F\varepsilon\sigma A(T^4 - T_\infty^4)$$

which we cannot solve analytically (we need to use a numerical solution). (Note: it is also somewhat questionable to use h_r in our *Bi* in this case).

OUTCOME:

Explain the utility of the Biot number

We can try to understand the meaning of the Biot number through mathematical means, by seeing where it *naturally* occurs....
Consider the boundary between a fluid and a solid.



Recall that the *boundary* is infinitely thin, and thus cannot hold any thermal energy. therefore, **even under unsteady conditions**, $q_{cond}=q_{conv}$, or

$$-k\frac{dT}{dx} = h(T - T_\infty)$$

If we generalize this boundary condition, by making it dimensionless, we need to define a characteristic length and temperature.
Using L as our characteristic length and T_∞ as our characteristic time, we get

$$-k\frac{d(\varphi T_\infty)}{d(\xi L)} = hT_\infty(\varphi - 1)$$

where φ is our dimensionless T and ξ is our dimensionless position. Taking our derivatives, we can pull out the constants to yield

$$-\frac{kT_\infty}{L}\frac{d\varphi}{d\xi} = hT_\infty(\varphi - 1)$$

which can be easily simplified to

$$\frac{d\varphi}{d\xi} = -\frac{hL}{k}(\varphi - 1) = -Bi(\varphi - 1)$$

We can then understand the "regimes" of transient response by asking ourselves what is the physical significance of small, large, and "order 1" Bi.
Clearly, small Bi means that the dimensionless temperature gradient must be small (i.e., there are no spatial variations in temperature), which is exactly what we reasoned before.
For order 1 Bi, we see that the dimensionless conduction and convection have roughly the same driving force (or, pessimistically, resistance).

NOTE:

In order to understand what we mean by things that are equal having different driving forces/resistances, consider running a three legged race with Yao. Who would be limiting your team's progress?! What does it physically mean for Bi>>1?!

OUTCOME:

Identify "regimes" of transient response based on the value of the Biot number

T: Solving Lumped Problems

If the Bi is determined to be sufficiently low in a transient problem, we can use the lumped equation to solve for the temperature evolution.

NOTE:

A good rule of thumb for something being bigger than 1 or smaller than 1 is that it varies by more than an order of magnitude. that is, 0.1 or less is smaller than 1 while 10 or more is greater than 1. In between, we have an "order 1" value.

In this case, we can write that the governing equation is

$$\frac{d(\rho c V T)}{dt} = \rho c V \frac{dT}{dt} = -hA(T - T_\infty)$$

As we will do with many problems moving forward, we would like to make this equation dimensionless prior to solving it. In order to do that we need to identify the relevant time and temperature scales (and length scales in problems where things vary by position). Recall that for this type of problem we will have **two** characteristic temperatures, the initial temperature T_o and the bulk fluid temperature T_∞.

For this reason, we use *both* temperatures to scale our variable by defining:

$$\theta = \frac{T - T_\infty}{T_o - T_\infty}$$

Plugging this into our equation yields

$$\frac{d\theta}{dt} = -\frac{hA}{\rho c V}\theta$$

where $\frac{\rho c V}{hA}$ clearly has units of time. Taking this as our characteristic time yields

$$\tau = \frac{thA}{\rho c V}$$

which can be plugged into our equation to give its dimensionless form

$$\frac{d\theta}{d\tau} = -\theta$$

This can be solved providing we recognize that our initial condition (i.e., $T=T_o$ at $t=0$) has now become $\theta=1$ at $\tau=0$) to yield

$$\theta = e^{-\tau}$$

NOTE:

This solution is much the same for linearized radiation or a combination of linearized radiation and convection as our external heat transfer mechanism(s), provided we replace our h in the τ definition.

OUTCOME:

Use the "lumped" equation to solve "1D" transient heat transfer problems

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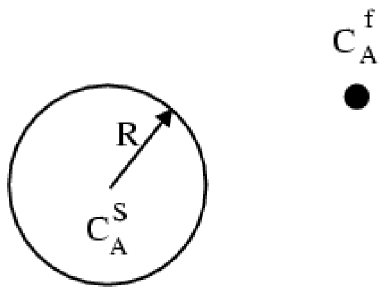
T: Lumped Mass Transfer Problems

Lumped mass transfer problems are quite similar to those of heat transfer.
Consider the case where a solid with some concentration of species A, C_A^S, is losing material to a bulk fluid.



If we focus on the interface between the phases, we could make the same argument that we did for the case of heat transfer to explain why the diffusive flux to the surface must be equal to the convective flux off of the surface:

$$-D\frac{dC_A^S}{dr} = k_c\left(C_A^f - C_A^{f\infty}\right)$$

where we need to note that there exists a "jump" in concentration at the interface (because at steady state the concentration in the solid and fluid phases would not be equal, but instead would be related via $mC_A^S = C_A^f$).
Using this relation and defining a fictitious solid concentration far away from the solid surface (i.e., the solid concentration that *would* be in equilibrium with the bulk fluid concentration), we get

$$-D\frac{dC_A^S}{dr} = mk_c\left(C_A^S - C_A^{S\infty}\right)$$

We could define a dimensionless concentration and spatial coordinate as

$$\varphi = \frac{C_A^S}{C_A^{S\infty}} \quad \xi = \frac{r}{R}$$

which can be plugged into our equation to give its dimensionless form

$$\frac{d\varphi}{d\xi} = -mBi_m(\varphi - 1)$$

where the Biot number is now defined (for mass transfer problems) as

$$Bi_m = \frac{k_c R}{D}$$

NOTE:

For consistency, we will continue to define the characteristic length in the Bi as V/A, so for a sphere it should have been R/3, above.

As with heat transfer, if Bi<<1, we get a "lumped" problem, whereby we can write the total rate of change of the mass/moles from our "point mass" as being equal to the external rate of mass flow:

$$\frac{d(C_A^S V)}{dt} = V\frac{dC_A^S}{dt} = -mk_c A\left(C_A^S - C_A^{S\infty}\right)$$

If we use both the bulk and initial concentrations to make C dimensionless (as we did in heat transfer), and notice that $V/(mk_cA)$ has units of time so that our dimensionless variables are:

$$\theta\frac{C_A^S - C_A^{S\infty}}{C_{A_o}^S - C_A^{S\infty}} \quad \tau = \frac{mk_c At}{V}$$

we get the same lumped equation that we did for heat transfer

$$\frac{d\theta}{d\tau} = -\theta$$

which, of course, yields the same solution

$$\theta = e^{-\tau}$$

OUTCOME:

Use the "lumped" equation to solve "1D" transient mass transfer problems

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T: Using Gurney-Lurie Charts

As we saw before, if we look at transient heat transfer problems in *dimensionless* form, we get a dimensionless temperature, a dimensionless time, and the Biot number out. Not surprisingly, if we have a problem whereby the Bi is **not** small, we also get a dimensionless position (since spatial gradients do not disappear).

One of the most *practical* reasons for making our equation (and therefore our answers!) dimensionless is that our results are then general. That is, since the same dimensionless equation arises for many differing dimensional systems, we can use the SAME dimensionless answer for all of them!

One particularly useful side-effect of this is that people have tabulated results of dimensionless solutions for transient problems.....we can use these for many problems as long as we know three of the four dimensionless parameters defining our system!

NOTE:

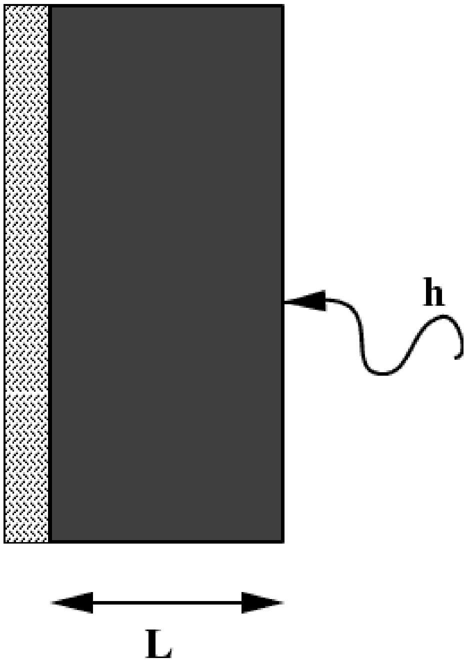
This approach works equally well for heat or mass transfer problems!

OUTCOME:

Use Gurney-Lurie charts to solve heat and mass transfer problems

TEST YOURSELF

Let's look at an example...



A flat wall of brick which is 0.5 m thick and originally at 200K has one side suddenly exposed to hot gas at 1200K. If the heat transfer coefficient on the hot side is 7.38W/m^2K and the other face is perfectly insulated determine

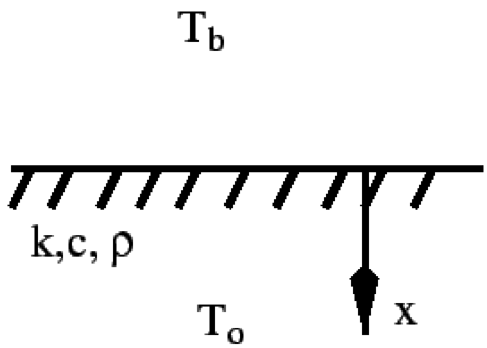
A. The time it takes to raise the center to 600K

B. The temperature of the insulated side of the wall at this time.

The properties of the brick are as follows....k=1.125 W/mK; c=919 J/kgK; ρ=2310 kg/m^3

T: Transient Conduction - Semi-Infinite

Consider conduction into the ground. The initial temperature of the ground is T_o and the air temperature (suddenly) drops from T_o to T_∞ at time, t=0. How do we find the temperature profile as a function of
position and time?



STEP 1:

Check what the value of the Biot number is ($Bi = \frac{hL}{k}$). What is L?

STEP 2:

Since the Biot number is **not** very small, we
cannot use "lumped". How about using the charts?!

In order to make our problem dimensionless, first we can choose θ to be

$$\theta = \frac{T - T_\infty}{T_o - T_\infty}$$

That is the easy part...

How do we make the length or time dimensionless? (since there is no obvious length or time scale!)

Last time, there was no obvious time scale, yet we made the problem dimensionless with a combination of the length scale and the properties of the material.

Clearly that won't work this time, but something similar DOES work...

What we do, based on the units, is define a new dimensionless variable which is a *combination* of *x* and *t*
(i.e., use our *variables* as the length and time scales!) so we get

$$\eta = \frac{x}{\sqrt{4\alpha t}}$$

where we stuck the 4 in for convenience...

Hey this gives us $T = F(\eta)$. This is a one dimensional problem!

Thinking about the original problem, we had an initial condition to satisfy: that T= T_o.
We also had two boundary conditions: that the surface temperature $T_s = T_\infty$ (because our Bi is effectively large) and that sufficiently far from the surface (i.e., at $x \to \infty$) the temperature has not changed yet so it is still T_o.

A little thought tells us that our new variable will have a hard time satisfying **all** of these conditions unless we have a third order ODE...

However, it turns out that the relevant differential equation (after plugging in our new variable) is:

$$-2\eta\frac{d\theta}{d\eta} = \frac{d^2\theta}{d\eta^2}$$

The good news is that even though we can only satisfy two conditions our boundary (initial) conditions become
$\theta = 1$ at $\eta \to \infty$; $\theta = 0$ at $\eta = 0$; and $\theta = 1$ at $\eta \to \infty$

NOTE:

> *Two of our boundary (initial) conditions, that used to be independent, are now the same! This is in fact critical for our answer to be possible (since we now have a second order ODE, we only need two conditions, so if we had three we couldn't guarantee to satisfy all of them!).*

This happens (and the problem's solution works) because this is what is called a self-similar or similarity problem (see here for a brief explanation of self-similarity.

In order to solve this equation, it is easiest to make the simple substitution that $p = \frac{d\theta}{d\eta}$, so that we get

$$-2\eta p = \frac{dp}{d\eta}$$

which we can easily rearrange and solve

$$\int -2\eta d\eta = \int\frac{dp}{p}$$

gives

$$-\eta^2 = \ln[p] + A$$

or

$$p = \frac{d\theta}{d\eta} = Be^{-\eta^2}$$

which we can integrate *again* to yield

$$\theta = \int_0^\eta Be^{-u^2}du + C$$

Using our BC's, $\theta = 0$ at $\eta = 0$ yields C=0 (since the integral from 0 to 0 is 0).
Our other BC $\theta = 1$ at $\eta \to \infty$ gives

$$1 = \int_0^\infty Be^{-u^2} du$$

where we can look up the fact that

$$\int_0^\infty Be^{-u^2} du = \frac{\sqrt{\pi}}{2}$$

so that

$$B = \frac{2}{\sqrt{\pi}},$$

which yields a final answer of

$$\theta = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-u^2} du$$

The integral in this expression cannot be solved analytically, but it shows up in so many problems that it has been given a name and values are easily found in tables. It is called the error function (*erf*)

$$erf[\eta] = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-u^2} du$$

So our final answer, given in its most compact form is

$$\theta = erf[\eta]$$

or

$$\frac{T - T_\infty}{T_o - T_\infty} = erf[\frac{x}{\sqrt{4\alpha t}}]$$

NOTE:

*This is an interesting result because, to quote Eddie Murphy in The Golden Child, "There's a ground Monty!". In other words, the problem really **doesn't** go to infinity, but as long as the thermal energy does not **know** that the ground is not infinite, this solution works. Taking this a step further, we note that $\sqrt{\alpha t}$ has units of* *length. This quantity can be thought of as the **penetration depth** or the distance over which thermal effects have penetrated during the time, t. So as long as our system size, L is much larger than the penetration depth, $L_p = \sqrt{\alpha t}$, we can use this solution.*

OUTCOME:

Use the semi-infinite approximation to solve both transient heat and mass problems for "short" times

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