Linear Transport Relations (LTR)

Much of Transport Phenomena deals with the exchange of momentum, mass, or heat between two (or many) objects. Often, the most mathematically simple way to consider how and how fast exchanges take place is to look at driving forces and resistances.

In momentum transport, we are interested in driving forces that arise from differences in pressure and/or velocity.

- Solve problems in fluid hydrostatics
  - Derive the pressure field equation [2.1] [notes]
  - Calculate the pressure distribution in a fluid or system of fluids that is at rest [2.2] [notes]
  - Use Archimedes' principle to calculate buoyant forces on (partially) immersed objects [2.3-2.4] [notes]
- Use friction factors and/or drag coefficients to calculate drag [12.2, 13.2, 13.3]
  - Distinguish between lift, drag, skin friction, and form drag[notes]
  - Calculate friction factors from correlations and read friction factors off of charts[notes]
  - Use friction factors and/or drag coefficients to calculate drag on submersed objects (external flows) [notes]
  - Estimate friction losses in pipes and pipe networks[notes]

In heat and mass transport, our driving forces arise from differences in concentration and temperature.

- Perform convection and convection/radiation problems
  - Perform convective heat transfer calculations [15.3, 19.1, 19.2] [notes]
  - Perform convective mass transfer calculations [24.3, 28.1, 28.2] [notes]
  - Perform radiative heat transfer calculations [15.4, 23.1, 23.2, (23.7)] [notes]
  - Calculate the thermal resistance and magnitude of heat flow in combined convective/radiative systems [15.5] [notes]
LTR: The Pressure Field Equation

If a fluid element is at rest, we can do a simple force balance on the element (recognizing that the sum of forces equals zero for a non-accelerating "body"): 

\[
0 = (P_1 - P_2)A + \rho g V
\]

dividing through by area leaves:

\[
(P_2 - P_1) = \rho gh
\]

If we let the distance, h, get small so that the change in pressure \((P_1 - P_2)\) also gets small, we get:

\[
dP = \rho g dz
\]

which we can rearrange to give:

\[
\frac{dP}{dz} = \rho g
\]

In three dimensions this gives us the Pressure Field Equation for a static fluid as:

\[
\nabla P = \rho \vec{g}
\]

**NOTE:**

*Here gravity is a vector pointing in the z direction.*

A more general form of this equation is possible if we relax the assumption that the fluid is not accelerating. In this case, we simply have the sum of the pressure and gravity forces equal to the mass of the fluid times acceleration:
\[ \rho V a = (P_1 - P_2)A + \rho g V \]

again dividing through by area leaves:

\[ \rho ha = (P_1 - P_2) + \rho gh \]

which for small \( h \) and \( P \) difference (in three dimensions) gives the general Pressure Field Equation:

\[ \nabla P = \rho (\ddot{g} - \dot{a}) \]

**OUTCOME:**

Derive the pressure field equation
Calculating the Pressure Field

In a static fluid the pressure field equation is given as:

\[ \nabla P = \rho g \]

In order to solve this we need to recognize the meaning of the \( \nabla \) (gradient) operator.

**DEFINITION:**

*The gradient is a measure of the rate of change of a quantity in space (a slope). It yields a vector pointing in the direction of maximum spatial rate of change, so that the gradient of the temperature, for example, is (in cartesian coords):*

\[
\nabla T = \frac{\partial T}{\partial x} \hat{e}_x + \frac{\partial T}{\partial y} \hat{e}_y + \frac{\partial T}{\partial z} \hat{e}_z
\]

Using this definition, and taking \( z \) to be in the vertical direction, the component form of the Pressure Field Equation is written as:

\[
\frac{\partial P}{\partial x} \hat{e}_x + \frac{\partial P}{\partial y} \hat{e}_y + \frac{\partial P}{\partial z} \hat{e}_z = \rho g \hat{e}_z
\]

Rearrange and simplifying gives that the pressure is constant in the x and y directions, but varies in the z direction as:

\[
\frac{dP}{dz} = \rho g
\]

\[
P_2 - P_1 = \rho g (z_2 - z_1) = \rho gh
\]

**NOTE:**

While this looks exactly like where we started with our balance of forces, it is actually different because it is true for the whole fluid continuum, not simply for the small control volume on which we were doing the balance (this will make more sense when you do the Test Yourself, below).

**OUTCOME:**

Calculate the pressure distribution in a fluid or system of fluids that is at rest

**TEST YOURSELF**

Calculate the pressure distribution in a static gas. What is different in this case. As an example approximate the pressure distribution in the Earth’s atmosphere.
LTR: Calculating the Pressure Field Example

OUTCOME:

Calculate the pressure distribution in a fluid or system of fluids that is at rest

TEST YOURSELF

Calculate the pressure distribution in a static gas. What is different in this case. As an example approximate the pressure distribution in the Earth’s atmosphere.

\[ \nabla P = \rho g \]

Taking this as a 1D problem in rectangular coordinates (close anyway) and putting the origin at the surface of the earth with positive \( z \) upward, we get:

\[ \frac{dP}{dz} = -\rho g \]

Assuming that air is an ideal gas, we can plug in \( PV = nRT \) where we solve for \( \rho = n/V \) so that \( \rho = P/RT \):

\[ \frac{dP}{dz} = -\frac{P}{RT} g \]

Rearranging and assuming that \( T \) does not change with \( z \) we get:

\[ \frac{dP}{P} = -\frac{g}{RT} dz \]

which we can integrate from the surface of the earth (at \( P = P_{atm} \)) to some atmospheric height, \( H \), where we will define \( P = P_0 \), to give:

\[ \int_{P_{atm}}^{P} \frac{dP}{P} = -\frac{g}{RT} \int_0^H dz \]

\[ P = P_{atm} e^{-\frac{gH}{RT}} \]
So far we have looked at the pressure and gravitational forces on a fluid element. What happens when we submerge a solid in the fluid?

**DEFINITION:**

*Archimedes' Principle* states that the buoyant force, the force resulting from the "subtracted" fluid (whose volume is $V_f$), is equal to the weight of the displaced fluid in the direction opposite to gravity:

$$\mathbf{F}_b = -\rho_f V_f \mathbf{g}$$

**NOTE:**

This force acts as a body force, going through the center of mass of the body, and therefore can be thought of as a density correction for the gravitational force (if the solid is completely submersed so that $V_f = V_s$):

$$F_g = (\rho_s - \rho_f)g V_s$$

**OUTCOME:**

Derive the pressure field equation

**TEST YOURSELF**

*Use Archimedes' principle to calculate buoyant forces on (partially) immersed objects: A block is floating with a fraction, $f$, of it under the water (whose density is $\rho$). What is the density of the solid?*
There are two forces acting on the block: the weight of the solid itself acting downward, the buoyant force (given as the weight of the displaced fluid). These forces must be equal if the block is not moving:

$$W = \rho_s g V_s = F_B = \rho_f g V_f$$

If we define $f$ as the fraction of the block that is under fluid, then $V_f = fV_s$ so:

$$\rho_s g V_s = \rho_f g (fV_s)$$

which can be rearranged to give:

$$\rho_s = f \rho_f$$